# DEVELOPMENT OF MATHEMATICAL IMAGINATION OF 3-DIMENSIONAL POLYHEDRA THROUGHOUT HISTORY AND INVERSION PHENOMENA
# 13TH INTERNATIONAL CONFERENCE ON GEOMETRY AND GRAPHICS

**Alexander HEINZ**
Herdecke, Germany

**ABSTRACT**: This presentation deals with the historical development of the mathematical imagination of 3-dimensional polyhedra and inversion phenomena. It is only possible to conceive the laws of geometrical forms objectively in thought. As a rule, these thoughts are formed through observation (description, illustration, or object) based on sensory or inner perception. The observer's perception is determined through his spatial standpoint in relation to the object he observes. Regular and semi-regular polyhedra are good examples of geometrically stable phenomena, introduced here as sensory objects in the form of descriptions, illustrations and as objects. The historical descriptions of polyhedra, examples and objects, are compared here from each spatial position of the observer towards the object under observation.

Different standpoints give one-sided and mutually opposing perceptions and interpretations of space (e.g. inner as opposed to outer). Only a comprehensive overview can show the totality of spatial forms of a polyhedron, thereby revealing a spatial structure. Examples from the history of architecture complete the comparisons. Plato's descriptions of regular solids and the Archimedean solids are compared with the accounts from the Renaissance (Leonardo da Vinci, Albrecht Dürer) and the discovery of the polar-Archimedean solids. An excursus on the history of architecture (Greek temple, basilica) completing the initial comparisions leads to a further comparison (pyramid, stone circle; or stone balls and a projection of regular polyedra).

In this second step, it is shown how polar-opposite ideas of space can be dynamically combined as a metamorphosis. If rhythmically developed, these metamorphoses appear cyclically. They can be shown as forms turned inside-out, through which, observing in thought, space is turned inside-out, or "inverted". With this at the same time a comprehensive concept of space is described. Paul Schatz was one of the first to recognise the importance of this, demonstrable through special mobile models of the Platonic solids. He discovered inversion, a completely new kinesmatic kind of movement, different from translation and rotation. At the same time, he gave the bases for some initial technical applications out of inversion, which are presented at the end (OLOID technology, turbula).

**Keywords:** regular polyhedra, history of human 3-D imagination, 3-D-inversion in engineering and science, Paul Schatz, inversion

……………………………………………………………………………………………………....

## 1.INTRODUCTION

Spatial consciousness is not a mathematical category, but a necessary prerequisite for comprehending mathematics and geometry. Geometry, though independent of space and time, can only be conceived in space and time. Consequently, it is relevant to speak of a historical development of geometrical discoveries. An aspect of this development are the presentations of regular polyhedra at various times in different cultural ages. Here I assume that a spatial consciousness corresponds to the respective presentation. If the special manner of a presentation depends on the degree of spatial consciousness, then retrospective appreciation of a spatial presentation rests on the spatial consciousness that comprehends it. If laws can be found in the historical development, these can shed light on present and future perspectives.

## 2.GREEK CIVILISATION AND THE RENAISSANCE

### 2.1Plato's approach, Archimedes

Plato (427–347 B.C.) was the first to describe the five regular polyhedra. In his dialogue *Timaeus* he argues how, out of two kinds of triangles, four polyhedra are formed, and a fifth is mentioned. In the same way the Archimedean solids can be formed, but for this two or three kinds of polygons are used. If this is recreated in the imagination, then all kinds of convex, regular and semi-regular polyhedra arise. Observer and object face each other. The object presenting its exterior to the observer is perceived as convex. With Plato and Archimedes (c. 287 to 212 B.C.) these presentations remain as descriptions.

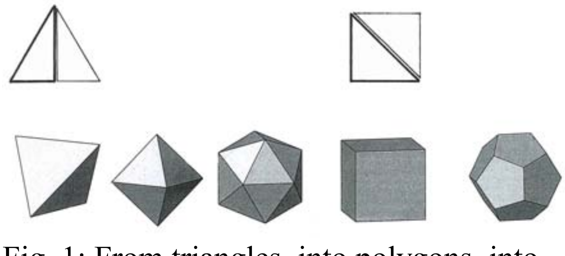

Fig. 1: From triangles, into polygons, into polyhedra; after Plato (*Timaeus*)

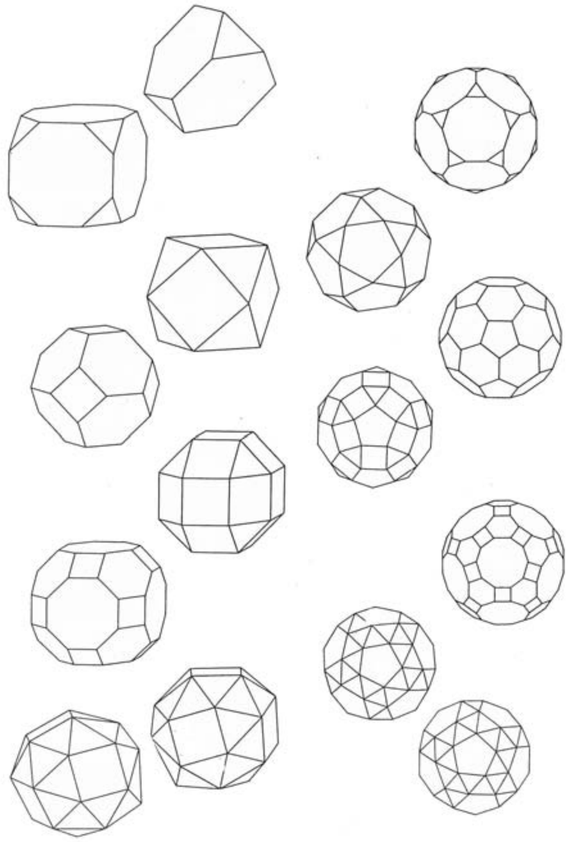

Fig. 2: Archimedean solids

### 2.2Rome

Originating in Roman times, bronze dodecahedra are known on which the corners are emphasised through ball-like elements; the centres of the surfaces are open. The observer can see into them, yet, because of their size, he/she remains outside the objects.

Fig. 3: Bronze dodecahedron, Rome

### 2.3Renaissance: Leonardo, Dürer

In Leonardo da Vinci's (1452–1519)

illustrations to Fra Luca Paciouli's (1445–post 1503) *Divina proportione* (1509), the regular polyhedra—along with others—are closed *and moreover* presented as models with edges, that is, open. In contrast to the Roman dodecahedra, Leonardos's dodecahedron with edges opens wider. The observer—in his imagination—can see through the polyhedron.

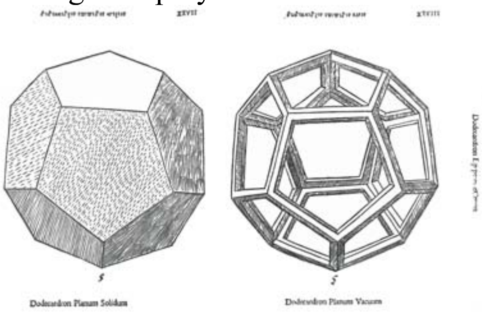

Fig. 4: Leonardo's dodecahedron; illustration for Pacioli's *Divina proportione*, 1509

In his book *Unterweisung der Messung* (1525) Albrecht Dürer (1471–1528) goes a step further. He draws the regular, and most of the semi-regular polyhedra on a plan, advising the reader or observer to trace the patterns and construct the models out of paper. If this is done, then as an observer one looks from inside the polyhedron. If large enough, it can entirely enclose the observer, who then can
experience it as concave, not as convex.

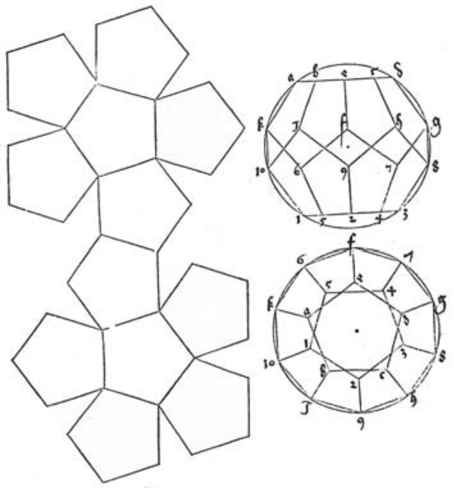

Fig. 5: Albrecht Dürer's dodecahedron plan, 1525

Consequently, according to the spatial standpoint of the observer, polyhedra can be experienced as convex spatial bodies as well as concave; the later perception demands to a higher degree his/her imaginative involvement. For the semi-regular solids a similar situation applies. The Archimedean solids are easy to construct out of polygons. The polar-Archimedean solids do not allow themselves to be constructed in this way without the exact knowledge of the respective lengths of the edges and relationships of the angles of the surfaces. An exact knowledge of the polar-Archimedean solids has to be acquired along with the concept of polarity. In this way Eugéne Charles Catalan (1814–94) described them completely for the first time.

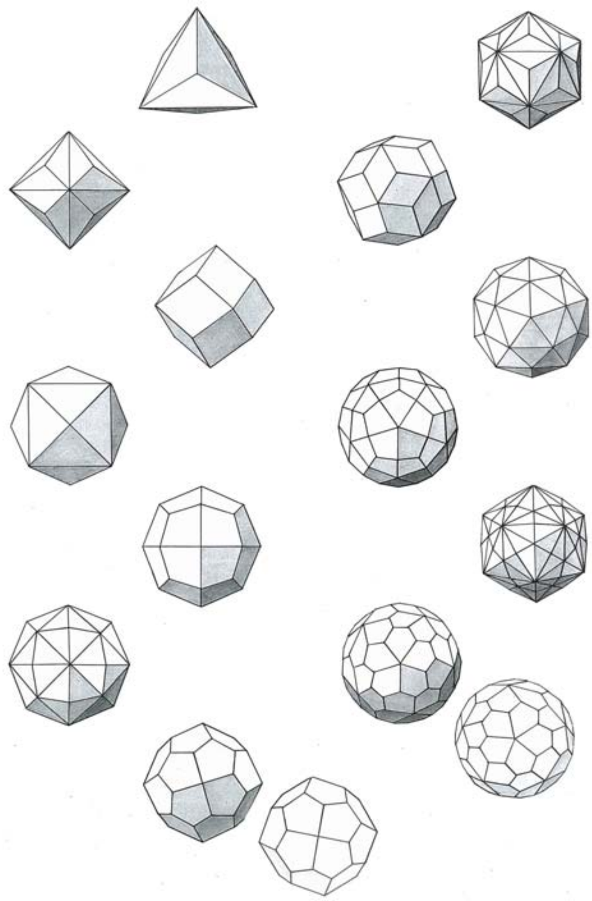

Fig. 6: Polar-Archimedean or Catalanian solids

Two things can be recognised in both these examples of “concave” and “polar”—they demand the ability to imagine spatially more strongly than the descriptions of either Plato or Archimedes. This corresponds to the thesis presented at the outset, of the development of the ability to conceive and think. On the other hand, the contrary standpoints of the two examples show polar relationships. The convex and concave conceptions face each other as earlier and later discoveries, in the same way as the discoveries of the Archimedean and polar-Archimedean were discovered earlier and later.

**2.4 Excursus on the history of architecture**
In a similar way basilica and Greek temple face each other. The temple has its pillars outside and its walls inside, which with the basilica is reversed.

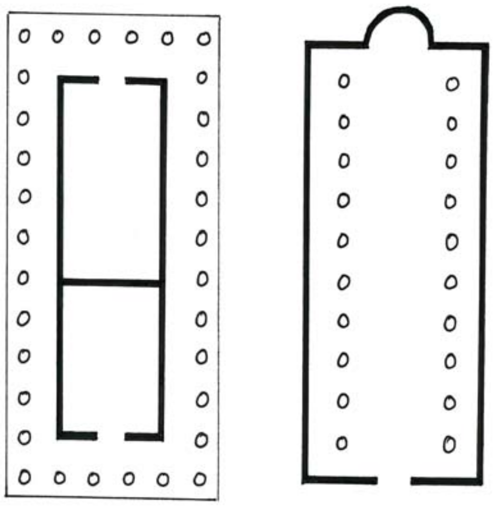

Fig. 7: Temple and basilica: contrasting principles

Frank Teichmann (1937–2006) taking up a suggestion of Rudolf Steiner (1861–1925) researched this phenomenon in detail in a comprehensive 4-volume work, the details of which cannot be discussed here. Here these examples serve only to gain a view of the relationship of the observer to his/her object. With temple and basilica inner and outer change places.

## 3.EGYPTIAN AND WESTERN EUROPEAN STONE AGE

**3.1 Cultural-historical aspects**
Before comparing temple and basilica, Teichmann compares Western European Stone Age and ancient Egyptian culture. He concludes that both cultures, which existed at approximately the same time, distinguished themselves in their diversity, as it were systematically. The pyramid is distinguished by its perfect geometrical construction, which draws the attention towards itself (centre). The stone circle, constructed for the main part with roughly hewn stones, serves observation of the heavens. It only receives its meaning through its arrangement in the landscape and to the starry constellations, drawing the attention of the observer to the distance (periphery).

The polar relationship between pyramid and stone circle as examples for the basic tone of the respective culture touch on the different directions into which the observer is drawn—with the pyramid to the inside, with the stone circle to the outside (these contrasts show many other aspects that cannot be discussed here; see Teichmann). It is worth noting here that the interior of the pyramid (centre), as well as the heavenly phenomena (periphery) remain unreachable for the observer.

With Teichmann, one can see the Greek temple is formed out of a well-balanced mixture of both elements of construction (pyramid and stone circle): the pillars are taken as if from the stone circle, and the inner sanctuary from the pyramid.

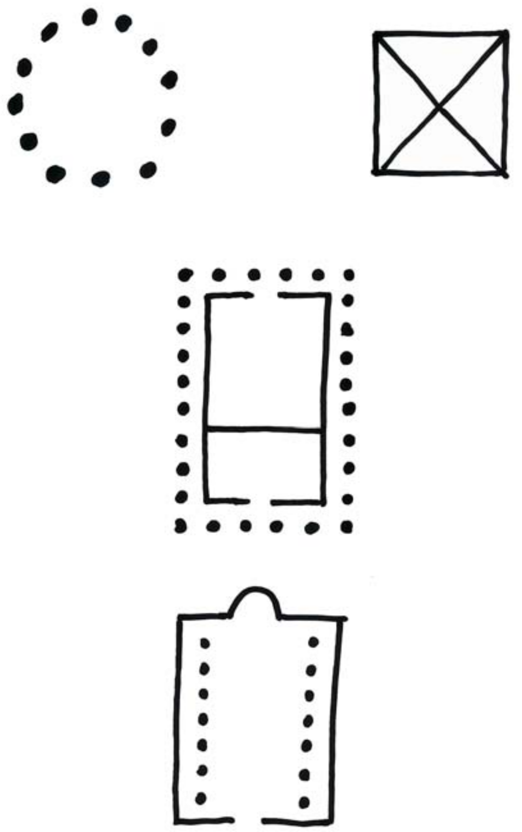

Fig. 7: Stone circle, pyramid; temple

**3.2 Carved stone balls**
In the British Isles, mainly in Scotland, carved stone balls were found which are distinguished through rich geometrical ornamentation. The balls were probably made c. 2,000 years before Plato. The forms of these balls show relationships to the regular, and other polyhedra. They appear to be composed of several small balls, which represent the corners or the surfaces of the related polyhedra.

Fig. 9: Carved stone balls, Scotland, c. 2,500 B.C.

These artifacts give the impression to the observer, as if the stonemason had not managed to complete the ball into a polyhedron. The forms appear as incomplete. The profusely carved amorphous ornamentation reinforces this impression. Similar to the placement of the stones, which points towards the periphery, the forces of form of the stone balls are not brought to completion in straight edges and flat surfaces.

**3.3 Metatron's cube**
A further presentation of the regular solids arises out of the projection of the regular polyhedra along the threefold axes of symmetry into a pattern of a sixfold series of circles, or through an infolded construction of the edges of the polyhedra. For this, particular points connect with each other. This method, with which all regular polyhedra can be constructed as projections, is also called Metatron's cube. The observer does not perceive a polyhedron, but only a projection. In its formal strictness, this presentation recalls the pyramids.

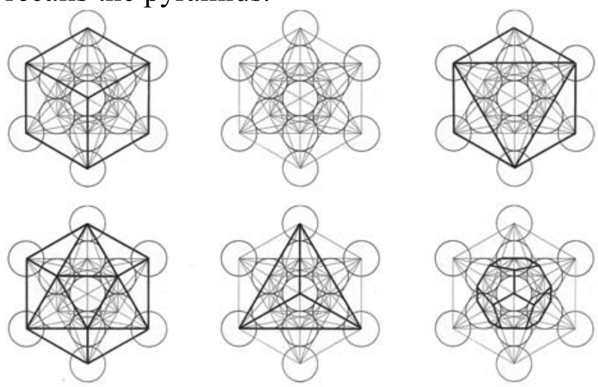

Fig. 10: Metatron's cube

**3.4 Polarities in comparing Egypt with the Stone Age, and Greece with the Renaissance**
In contrasting the representative buildings of Egypt with the Stone Age and Greece with the West, parallels can be drawn to the contrasting of Metatron's cube with the stone balls, and the Platonic solids with the presentations of Leonardo and Dürer.

Neither with the stone balls nor with Metatron's cube do we reach the polyhedra with straight edges, pointed corners and level surfaces. Both emphasise different aspects of space, which on their own do not form a clear spatial structure. The stone balls emphasise the outer, the projections the centre. Only in the polyhedra of Plato and Archimedes do both meet to form a clear spatial structure.

This is the prerequisite for the laying hold of further thoughts in which opposites mutually depend, e.g. the polar relationships of the regular polyhedra to each other in the respective number of corners and surfaces, or the relationships of the Archimedean to the polar-Archimedean solids.

## 4.METAMORPHOSES

### 4.1 Polar levels and their transitions

These relationships turn stage by stage into their opposite when a regular polyhedron, through depressing the corners and buckling the surfaces, step by step is changed via Archimedean (or polar-Archimedean) solids into its polar-opposite solid.

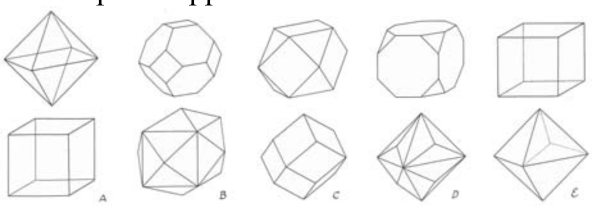

Fig. 11: Buckling and depressing regular polyhedra (hexahedron and octahedron)

### 4.2 Dynamic metamorphoses between poles

Every regular, even semi-regular solid can be changed into its polar opposite. First a simple observation. When a force moves towards a point, where the point offers no resistance, the force continues in the same direction beyond the point, leaving it as soon as it has reached it.

Figur 12: A force of direction passing through a point

In a similar manner one can observe the surfaces of any polyhedron as formed by forces which are effective from outside, and the corners as the counter-force effective from inside. If the force increases equally from all sides, the polyheadron becomes first smaller and finally a point. If the forces continue undiminished in the same direction, instead of forming surfaces from the outside they form corners from the inside. A polyhedron comes about which is the polar opposite of the initial polyhedron.

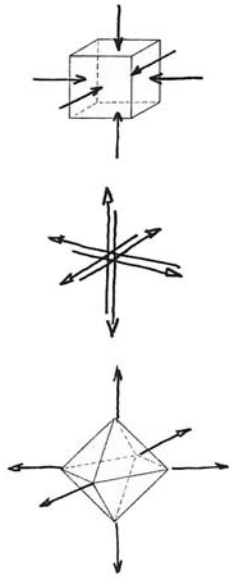

Fig. 13: Metamorphosis of directions of force in going through a point (hexahedron and octahedron)

The important thing with this example is the sudden change of the one thing into the other. This comes about through the dynamic of the whole process in going through the zero-point. The concept of infinity in its smallest form, the point, here marks an essential turning point of metamorphosis.

In a similar manner one can think the transition of the temple to the basilica, or the change in imagining a (merely) convex to an (also) concave polyhedron.

## 5.INVERSION—TURNING INSIDE OUT

### 5.1 Paul Schatz

Paul Schatz (1898–1979) was one of the first to recognise the importance of the theme

*Umstülpung*, inversion, or more accurately, a turning inside out. He researched it comprehensively as a dynamic method through which the infinitely small, rhythmically recurring, can be changed into the infinitely large—and vice versa. He demonstrated how space entire turns completely inside out, that is, its inner side turns outwards and vice versa.

**5.2 Inversion of the cube and of space**

First Schatz discovered that the cube as a (projective) complete polyhedron is able to be turned inside out, and later that the principle applies to every polyhedron. Through the model of Schatz's cube, one can show very well the (projective) inversion of the entire space—the planes of the cube tip sideways through six mobile joints until they reach a level prior to reforming, this time to a hollow cube. The change has taken place from inside to outside. Through further development the cube returns to its original form, following a cyclical and rhythmical movement of inversion.

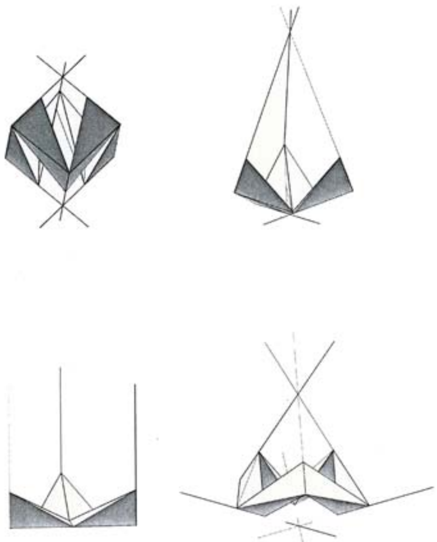

Fig. 14: Inversion of space through the example of a cube

**5.3 Platonic inversion**

All regular polyhedra can be inverted, that is, turned inside out. A special form of inversion was discovered by Immo (*1936), Franz (*1935) and Friedemann Sykora (*1961). In addition to the complete inversion of space it was discovered that all geometrical elements of a regular polyhedron can be turned inside out at the same time—corners, edges, surfaces and volume. Konrad Schneider (*1954), Wolfgang Maas (*1954) and Robert Byrnes (*1944) were involved in this research and made some discoveries. With the example of the invertible cube of Schneider, the above-described process is demonstrated. This inversion, with reference to the perfection of Platonic solids, is described as the Platonic inversion.

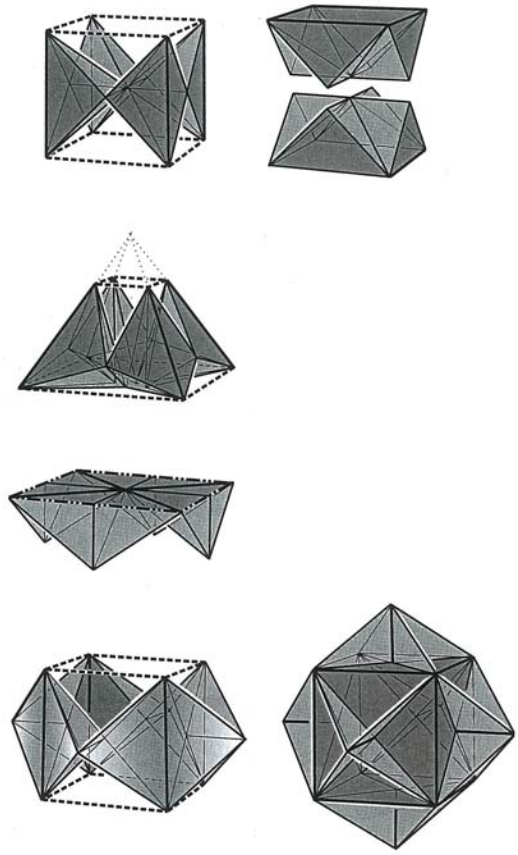

Fig. 15: Platonic inversion—invertible cube of Konrad Schneider

## 6 TECHNOLOGICAL APPLICATION AND USE

**6.1 Linkages**

Schatz's cube can be technically described as a sixfold linkage, which possesses one degree of freedom. The mobility motion of this linkage is consequently only possible in two directions—forwards and backwards.

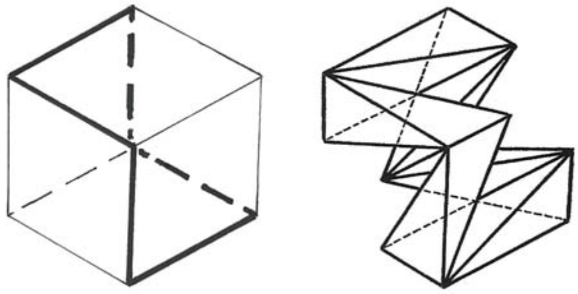

Fig. 16: Sixfold linkage in the cube

As the basis for a technological application of the inversion, only such polyhedra-inversions are suitable. Klaus Ernhofer (*1962) found a total of twelve possibilities to invert regular polyhedra with a sixfold chain of joints. He also developed the prototype of a machine (pulsina) whose inversion-movements derive from the inversion of the dodecahedron.

**6.2 Inversion—a new kinematic kind of movement**

From the discovery of inversion, Paul Schatz found a new kinematic kind of movement, the inversion (translation and rotation are already known). This kind of movement is a looping, pulsing movement in space which, with processes of acceleration and deceleration, always rhythmically returns to its starting point. Technically this can be applied where rotating starting energy is transferred through a corresponding mechanism into inversion movement.

**6.3 Turbula technology**

With the turbula and similar machines (holdyna, inversina), a cylindrical container is inversively moved through two axes rotating in opposite directions. Through two mobile arms the rotating is translated into an inverse movement. It is used in the mixing of fluid and solid matter, e.g. paint and medicines.

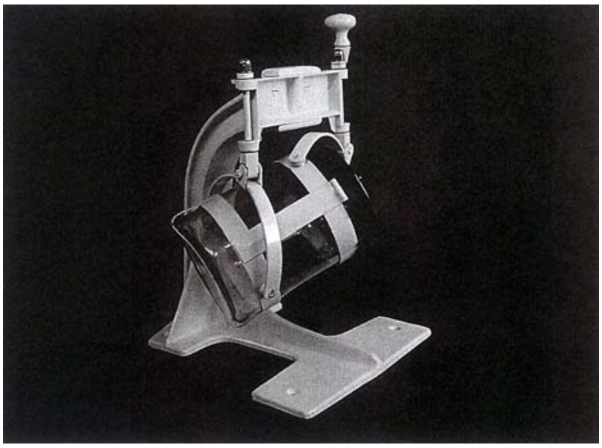

Fig. 17: Turbula

**6.4 Oloid and oloid technology**

The form of the oloid can be directly gained out of the inversion of Schatz's cube. It is similar to the windscreen-wiper which as a space-time surface creates a free surface on a windscreen for the view of the driver. The surface of the oloid is formed through the path taken by the diagonal of Schatz's cube moving through space in an inversion-cycle. The start and transmission of the turbula and the oloid are in principle the same. In contrast to the turbula, the oloid moves the surrounding space. The oloid is used as a aerator over water-surfaces or as an agitator under the surface of water. Oloid technology is especially used in treating stagnant water, in sewage works, and in aquariums.

Fig. 18a: Oloid

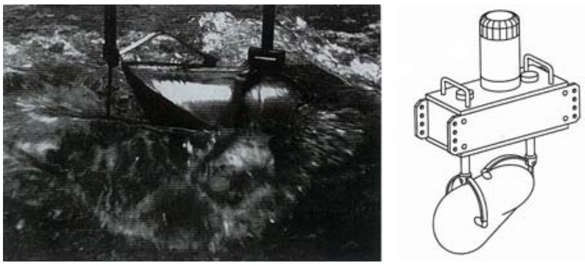

Fig. 18b: Oloid in use

**6.5 Advantages in its use**

Turbula technology as well as oloid technology have proved to be homogenous, thorough and non-destructive. The special efficiency (compared to rotational technical alternatives, c. 6/7 saving of energy is possible) of this technology is due to the special rhythmical manner of movement which is very similar to the movement of flowing water and that of fishes.

## 7 CONCLUSIONS

With the example of the various historical presentations it was shown in what manner perceptions and conceptions of space depend on the spatial relationship of the observer to his/her object. The more flexibly the observer thinks and imagines, the more comprehensive is his/her relationship. In the inversion, or turning inside out, the maximum of flexibility is connected with a geometrically strictly lawful and systematic sequence to change the standpoint and with it completely to lay hold of space in its infinite aspect.

Looking at the development of the ability to understand space chronologically, as has been presented, it is possible to recognise in a historical development how inversion can be grasped out of the individual aspects of space and their metamorphoses. Furthermore, one can see that only inversion contains all the geometrical elements through which the whole of space can be taken hold.
That this knowledge is the starting point of a completely new technology based on the movement of inversion, emphasises the fact that conscious and reflective perception ultimately not only determines conceptualising and thinking but also action. The discovery of inversion is a very recent one in the geometrical field as well as the technological field. Consequently, we may look forward to future technological developments, and I would like to invite you to take part actively or passively in further research.

All the models, illustrations and machines mentioned in the text will be shown in a demonstration.

## ACKNOWLEDGMENTS

Many thanks for the translation and helpful suggestions: Alan Stott, Jürgen Peters, Dr. Robert Byrnes.

## REFERENCES

1. Plato. Timaeus. 53C
2. Luca Pacioli. Divina proportione. Illustrationen von Leonardo da Vinci. 1509. Reprint France 1988.
3. Albrecht Dürer. Unterweisung der Messung mit dem Zirkel und Richtscheidt. 1525. Reprint Nördlingen, 2000.
4. Marshall, Dorothy N. Carved Stone Balls. Proceedings of the Society of Antiquaries of Scotland 108, Pps.4-72, 1983
5. Drunvalo Melchizedek. The Ancient Secret of the Flower of Life. 1994.
6. Teichmann, Frank: Der Mensch und sein Tempel. Vol. 1: Ägypten, Vol. 2: Griechenland, Vol. 3: Megalithkultur, 1983, Vol. 4: Chartres. Stuttgart.
7. Schatz, Paul. Rhythmusforschung und Technik. Stuttgart, 1998.
8. Maas, Wolfgang und Sykora, Immo. Umstülpungsmodelle der Platonischen Körper. Berlin, 1993.
9. Ernhofer, Klaus und Maas, Wolfgang. Umstülpbare Modelle der Platonischen Körper. Dornach, 2000.
10. Adam, Paul und Wyss, Arnold. Platonische und Archimedische Körper. Bern und Stuttgart, 1994.
11. Arn, Walter: Auf zu den Sternen. Hölstein, 2000.
12. Ziegler, Renatus. Platonische Körper. Dornach, 2003.
13. Byrnes, Robert. Das Riegelmodell von Paul Schatz. Mensch und Architektur 60, Berlin, p 82, 2007.

## ABOUT THE AUTHOR

Alexander Heinz has a practical approach to geometry by modelling and working in educational projects based on experience. His main interest are polyhedra and inversion phenomena. He cooperates with others e.g. Paul-Schatz-Stiftung, Basel, in developing didactical concepts that advocates practical experiences of mathematical laws by modelling with paper, woods and clay in an aesthetic way. He is a bookbinder. In his bookbindery he develops and produces movable geometric models and handcrafted books.